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TORSIONAL VIBRATION OF A RIGID CIRCULAR BODY ON AN INFINITE ELASTIC STRATUM

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Abstract—Exact formulation of this mixed boundary-value problem is presented by establishing the governing dual integral equations for the usually assumed unknown dynamic stress distribution under the rigid body. A kernel of these equations contains the hyperbolic tangent thereby making an exact solution formidable. However, the function $x/(1+x^2)^{\frac{1}{2}}$ has been found to be a good approximation for tanh x for all real values of x and its introduction leads to the conclusion that an elastic stratum excited by a torsional oscillator at a frequency factor η_2 behaves approximately as a semi-infinite medium excited at an equivalent frequency factor $\eta_{2e} = (\eta_2^2 - 1/\bar{h}^2)^{\frac{1}{2}}$ where $\bar{h}$ is the depth of the stratum referred to the oscillator radius and $1/\bar{h} \leqslant \eta_2$. Resonance curves for the rigid body based on this new approach are in good agreement with published experimental results.

1. INTRODUCTION

SO MUCH has been discussed in the literature since the time of Boussinesque on the mixed boundary-value problem of the identation of an elastic half-space or semi-infinite medium by a rigid body that there is a tendency to forget the more practical medium—the elastic stratum. The problem is simply summarized: within what limits does a medium of finite depth behave as a semi-infinite medium?

This question needs to be answered when considering the vibrations of buildings, dams, towers and similar large structures on soils where test holes and other geological surveys show clearly that soil of fairly uniform composition cannot, in general, be found to any depth comparable with the rather large dimensions of the foundations of the structures. Any theory based on the semi-infinite medium model either for calculating the response of these structures when subjected to external excitation or for estimating the dynamic elastic properties of soil from resonance tests as given, for example, by the author [1, 2] is bound to be vitiated by the lack of homogeneity of the soil to any depth which can reasonably be regarded as semi-infinite. On the other hand, a division of the sub-soil into strata of uniform elastic properties is a more realistic model. It is reasonable to suppose, as a first approximation, that the effect of the first layer or stratum is dominant and this layer can be regarded as being fixed at its base to the next layer which is usually more rigid and can be presumed, as a first step, to be unaffected by vibrations emanating from the free surface of the first layer. The problem is further simplified by proposing that the stratum is infinitely wide so that reflection of waves from edge boundaries can be neglected.

An attempt to answer this question was first made by Arnold *et al.* [3] who considered both the vertical and torsional vibrations of a rigid circular body on an infinite elastic stratum. Further work on the problem of vertical vibration was done later by Warburton [4]. Their approach of first assuming the unknown stress distribution under the rigid body has been discussed by Awojobi and Grootenhuis [5]. It is noted here in addition that, to assume as a starting point that the stress distribution under a *vibrating* rigid body on a medium of *finite* depth is the same as that under a *static* rigid body on a medium of *infinite* depth is clearly a first approximation remote from the real situation especially for vibrations of frequency factor greater than unity and for bodies whose dimensions are comparable with the stratum depth. This fundamental assumption is likely to have contributed to the doubtful result obtained by Warburton [4] where his curve for the variation of mass ratio with resonant frequency factor in the case of the semi-infinite medium lies between finite media curves (Fig. 7 of his paper). This strange result suggests, for a given rigid body and given medium, that there exists a turning point in the behaviour of the stratum as its depth is increased—an improbable result, the physical meaning of which was not mentioned in that paper.

The present work considers a new approach towards giving a fair answer to the above question by treating the case of a rigid circular body performing harmonic torsional oscillation on an infinite elastic stratum. The author holds that a legitimate procedure is to avoid assuming from the outset the form of an unknown stress distribution but to reformulate the problem in terms of dual integral equations, the solution of which would give the dynamic stress distribution under the rigid body and, indeed, any stress or displacement function throughout the entire stratum can be readily computed from the appropriate integral representation which contains the Hankel transform of the stress distribution under the rigid body as the only unknown.

It is shown in the next section that the first of the dual integral equations—established for the first time—involves a hyperbolic tangent. An exact solution is, at present, impossible.

2. DEVELOPMENT OF THE GOVERNING DUAL INTEGRAL EQUATIONS

In order to avoid repetition of previous work, the paper by Awojobi and Grootenhuis [5] shall, hereafter, be referred to as paper 1.

It has been shown in paper 1 that an isotropic elastic medium excited by a rigid circular torsional oscillator at its boundary is in a state of pure shear. Every particle in the medium vibrates in a small circular arc with a tangential displacement v, from the mean position, given by the one-dimensional equation of motion

$$\frac{\partial^2 v}{\partial z^2}+\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv)\right\}+\Omega_2^2 v = 0 \tag{1}$$

in which r, z are the independent variables of cylindrical coordinates and $\Omega_2 = \Omega/c_2$ where Ω is the angular frequency of excitation, c_2 is the velocity of waves of distortion in the medium and it is understood that v varies harmonically with respect to time i.e. $v = v_0 e^{i\Omega t}$. Equation (1) can be reduced to the second order ordinary linear differential equation

$$d^2\bar{v}(p, z)/dz^2 - \alpha_2^2\bar{v}(p, z) = 0 \tag{2}$$

where $\bar{v}(p, z)$ is the first order Hankel transform of $v(r, z)$ and α_2 is the double-valued function of the transform parameter, p given by

$$\alpha_2 = (p^2 - \Omega_2^2)^{\frac{1}{2}}$$

The general solution of equation (2) is

$$\bar{v}(p, z) = Ae^{-\alpha_2 z} + Be^{+\alpha_2 z} \tag{3}$$

where A and B are arbitrary functions of p to be determined from the boundary conditions. It should be noted that it is necessary to include in equation (3)—which is the equivalent of equation (11) of paper 1—the exponential term of positive index since all values of z are finite in this case and $\bar{v}$ remains bounded throughout the medium.

The mixed boundary conditions at the surface are

$$\left.\begin{aligned} \theta &= \theta_0 e^{i\Omega t} && (0 < r < R), \\ \tau_{z\theta} &= 0 && (r > R), \end{aligned}\right\} \tag{4}$$

where θ_0 is the angular amplitude of the harmonic oscillation of the rigid body and they can be embraced in the single condition

$$\tau_{z\theta} = \tau(r) \qquad (z = 0, r > 0) \tag{5}$$

where $\tau(r)$ is the unknown stress distribution to cause a constant angular displacement under the rigid body. We notice that $\tau(r)$ is a discontinuous function of r valid throughout the surface and also varies harmonically with respect to time i.e. $\tau(r) = \tau_0(r)e^{i\Omega t}$ where $\tau_0(r)$ is the stress amplitude.

We shall assume that there is no motion at the base of the stratum i.e. the base is rigidly fixed to a foundation, giving

$$v = 0 \qquad (z = h, r > 0) \tag{6}$$

where h is the depth of the stratum.

Remembering that the shear stress $\tau_{z\theta}$ is related to the displacement v by the equation of elasticity

$$\tau_{z\theta} = G\frac{\partial v}{\partial z} \tag{7}$$

where G is the medium shear modulus, and by taking the transforms of equations (6) and (7), substituting in the general solution of equation (3), we have the following simultaneous equations for evaluating the arbitrary functions A and B:

$$\left.\begin{aligned} -G\alpha_2(A-B) &= \overline{\tau(r)} \\ Ae^{-\alpha_2 h} + Be^{+\alpha_2 h} &= 0 \end{aligned}\right\} \tag{8}$$

from which it is easily shown that

$$A = -[e^{\alpha_2 h}\overline{\tau(r)}]/[2G\alpha_2 ch(\alpha_2 h)]$$

$$B = +[e^{-\alpha_2 h}\overline{\tau(r)}]/[2G\alpha_2 ch(\alpha_2 h)]$$

where $\overline{\tau(r)}$ is the first order Hankel transform of $\tau(r)$.

In order to formulate the representative dual integral equations, we only need now to return to the exact boundary conditions at the surface given by equations (4). It is convenient to re-express the first of equations (4) as

$$v = r\theta, \qquad (0 < r < R) \tag{9}$$

and apply Hankel's inversion theorem—equations (6a), paper 1—to recover v from equation (3). Similarly, the second of equations (4) can be written in an integral form to express the condition that the domain at the surface away from the rigid body is stress free.

This leads, after a few reductions, to the governing dual integral equations:

$$\left.\begin{aligned} \int_0^\infty \frac{\tanh(\alpha_2 h)}{\alpha_2}\overline{\tau(r)}pJ_1(pr)\,\mathrm{d}p &= -G\theta r \qquad (0 < r < R) \\ \int_0^\infty \overline{\tau(r)}pJ_1(pr)\,\mathrm{d}p &= 0 \qquad (r > R) \end{aligned}\right\} \tag{10a}$$

which are derived for the first time to give an exact formulation of the mixed boundary-value problem.

It is convenient to introduce the following non-dimensional quantities

$$\tilde{r} = \frac{r}{R}, \qquad \tilde{h} = \frac{h}{R}, \qquad \eta = pR, \qquad \eta_2 = \Omega_2 R, \qquad \alpha_2(\eta) = (\eta^2 - \eta_2^2)^{\frac{1}{2}}$$

and

$$F(\eta) = p\overline{\tau(r)}$$

and the dimensional quantity:

$$c = -GR\theta,$$

The non-dimensional form of equations (10a) is then

$$\begin{aligned} \int_0^\infty \frac{\tanh(\alpha_2 \tilde{h})}{\alpha_2} F(\eta)J_1(\eta\tilde{r})\,\mathrm{d}\eta &= c\tilde{r} \qquad (0 < \tilde{r} < 1) \\ \int_0^\infty F(\eta)J_1(\eta\tilde{r})\,\mathrm{d}\eta &= 0 \qquad (\tilde{r} > 1). \end{aligned} \tag{10b}$$

It should be noted if we had assumed, instead of equation (6), the alternative boundary condition

$$\tau_{z\theta} = 0 \qquad (z = h, r > 0) \tag{11}$$

which regards contact between the base of the stratum and the foundation to be frictionless, we find that the sole effect on the pairs of equations (10) is to change the hyperbolic tangent to hyperbolic cotangent.

3. A USEFUL APPROXIMATION TO THE HYPERBOLIC TANGENT

It has been the practice to obtain approximate solution to a problem involving the hyperbolic tangent, either by making the crude assumption $\tanh x = 1$ or carrying out the tedious procedure of expressing it as an infinite series of exponentials with negative indices, i.e.

$$\tanh x = (1 - e^{-2x})/(1 + e^{-2x})$$

which, on expanding as a geometric series, results in

$$\tanh x = 1-2\sum_{n=1}^{\infty}[(-1)^{n-1}e^{-2nx}] \qquad \text{for all } x > 0.$$

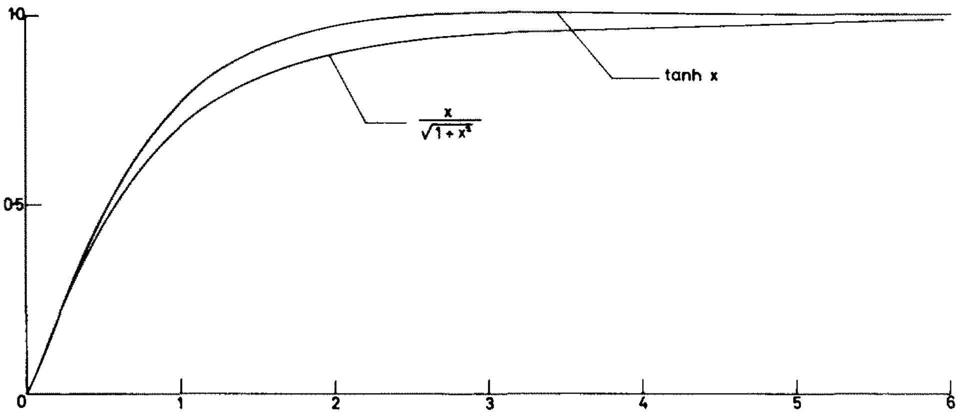

FIG. 1. A useful approximation to the hyperbolic tangent.

There are no known methods of obtaining an exact solution of the pairs of equations (10) developed above. The method of successive approximations presented in paper 1 cannot as yet be used unless the kernel containing the hyperbolic tangent can be expressed as a series in the form which is amenable to the treatment of the first approximation solution discussed in Section 7 of paper 1. Unfortunately, no asymptotic expansion is known for the hyperbolic tangent or, indeed, for any other exponential function. The geometric series given above is too cumbersome to be useful in this type of problem.

The above considerations have led the author to appreciate the usefulness of the function $y = x/(1+x^2)^{\frac{1}{2}}$ which has been discovered to be adequately close to the hyperbolic tangent as shown in Fig. 1 even with the vertical scale 2·5 times as large as the horizontal. It is to be noted that the two functions and their derivatives are identical at the origin and at infinity. Also the numerator, x, can be regarded as $\sinh x$ to make the denominator $\cosh x$, but while $\sinh x \doteqdot x$ is valid only for value of x much less than unity, the above approximate function holds for all real values of x.

4. SOLUTION OF THE DUAL INTEGRAL EQUATIONS

We are now in a position to consider the solution of the dual integral equations (10b). Replacing the hyperbolic tangent by the function discussed above and simplifying, we reduce equations (10b) to

$$\left.\begin{aligned} &\int_0^{\infty}\frac{F(\eta)J_1(\eta\tilde{r})\,d\eta}{(\eta^2-\eta_{2e}^2)^{\frac{1}{2}}} = c\tilde{r} \qquad (0<\tilde{r}<1) \\ &\int_0^{\infty}F(\eta)J_1(\eta\tilde{r})\,d\eta = 0 \qquad (\tilde{r}>1) \end{aligned}\right\} \tag{12}$$

where $\eta_{2e}^2 = \eta_2^2 - 1/\bar{h}^2$ defines an equivalent frequency factor (equation (37a), paper 1) in relation to the depth of the stratum.

This leads us directly to the important conclusion that every result of the stratum problem can be derived from the corresponding semi-infinite medium result simply by modifying the dynamic stress distribution under the rigid body in the semi-infinite medium case—a function of η_2—to the same function of η_{2e} to obtain the dynamic stress distribution under the rigid body in the stratum case. Since the dynamic stress distribution in each case is, strictly speaking, the only unknown in the problem, the modified stress distribution can now be used in the appropriate integral representation of any stress or displacement function that is required in the entire stratum.

The pair of equations (12) has been solved in paper (1) and it is only left here to illustrate the degree of fairness of the present approximate method of deducing the solution of the stratum problem from the semi-infinite result. This is done by considering a case for which experimental results are available—the amplitude response of the rigid torsional oscillator on an elastic stratum. We shall consider cases for which η_{2e} is positive for the obvious reason that this is the case for which direct deduction from the semi-infinite solution is valid. It is to be noted, however, that this case covers most practical cases of frequency factor η_2 provided the stratum depth is greater than the radius of the oscillator i.e. $\bar{h} > 1$. If, however, we are interested in negative values of η_{2e} we only need to consider contours of integration in the manner of paper (1) in which the only pole of the integrand within the contour is on the imaginary axis.

Reverting to the solution of the amplitude response of the rigid body on the elastic stratum, the equation of motion of the rigid body is written with the modified dynamic stress distribution integrated over the circular domain of the oscillator. We then arrive at the non-dimensional amplitude $\hat{\bar{\theta}}$ of the oscillator after the manner of equation (66) of paper 1 i.e.

$$\left.\begin{aligned} \hat{\bar{\theta}} &= (GR^3\hat{\theta})/\hat{T} = 1/(P+iQ) \\ |\hat{\bar{\theta}}| &= 1/\sqrt{(P^2+Q^2)} \end{aligned}\right\} \tag{13}$$

where the functions P and Q have been computed in the present case up to the third approximation, $F_{33}(\eta)$ in accordance with the systematized scheme of successive approximations in paper 1. Again, P is the function accounting for the inertia of the oscillator and the stiffness of the medium while Q is a damping function generated by the dispersion of elastic waves in the medium. The results of computation give

$$\begin{aligned} P &= \tfrac{16}{3} - (\bar{J}\eta_2^2 + \tfrac{8}{45}\eta_{2e}^2) - 0{\cdot}89\eta_{2e}^2(1 - 0{\cdot}63\eta_{2e}^2 + 0{\cdot}305\eta_{2e}^4 - 0{\cdot}083\eta_{2e}^6 + 0{\cdot}0122\eta_{2e}^8) \\ Q &= 0{\cdot}755\eta_{2e}^3(1 - 0{\cdot}56\eta_{2e}^2 + 0{\cdot}216\eta_{2e}^4 - 0{\cdot}040\eta_{2e}^6 + 0{\cdot}0027\eta_{2e}^8). \end{aligned} \tag{14}$$

We notice that the inertia term $\bar{J}\eta_2^2$ in function P has η_2^2 and not η_{2e}^2 for the obvious reason that the inertia of the body does not depend on the stratum depth but only on frequency of excitation.

It should be noted that the expressions for P and Q in the above equation (14) differ from the corresponding expressions for the semi-infinite medium given on top of page 58 of paper 1 because the expressions here are given to third order approximation whilst those of paper 1 are to the eighth order.

The difference between the two expressions is not significant when the frequency factor η_2 is less than unity. For example, for a body of $\bar{J} = 3{\cdot}49$ on the semi-infinite medium and $\eta_2 = 1$ the above expressions give $|\hat{\bar{\theta}}| = 0{\cdot}83$ whilst paper 1 gives 0·82.

For higher values of frequency factor, the eighth order approximation in paper 1 has to be extended to include all higher powers of η_2 contributed by preceding orders of approximation since the contributions of such terms cannot be regarded as small compared with the terms retained. The alternative procedure is to calculate for each frequency factor the numerical value of the first order approximation and use this to generate successive approximations. This, however, does not possess the elegance of obtaining an analytical expression.

On the other hand, the expressions in equation (14) are within 2 per cent error of a point by point numerical procedure up to the third approximation for $\eta_{2e} \leq 2$ because the higher powers of η_{2e} not included are indeed negligible.

5. DISCUSSION OF THEORETICAL RESULTS AND COMPARISON WITH EXPERIMENTS

Theoretical resonance curves have been plotted in Figs. 2 and 3 using equation (13) with the derived functions P and Q. The effect of stratum depth can be studied from curves like those in Fig. 2 which shows the response of a given rigid body i.e. inertia ratio, on strata of increasing depth. An expected trend of asymptotic approach of the stratum case to the semi-infinite medium case is clearly seen and, for this inertia ratio, one would take a stratum depth of five times the oscillator radius as a fair approximation to the semi-infinite medium. Experimental results due to Arnold *et al.* [3] for the same inertia ratio on a medium that is practically semi-infinite have been included for the purpose of comparing the limiting resonance curve, $\tilde{h} = \infty$. The decrease of the resonant frequency factor with increase in stratum depth confirms the expected result that increase in stratum depth would give rise to a greater dispersion of elastic waves and, therefore, of "dispersion damping". It is significant that, for this inertia ratio, there is only a slight reduction in resonant frequency factor while amplitude is disproportionately reduced. This is explained by an inspection of the function P which shows that for inertia ratios of magnitude comparable with or higher than the static stress distribution term i.e. the leading term $\frac{16}{3}$, resonant frequency factor is mainly affected by the inertia ratio and not the stratum depth but the amplitude at resonance is governed by both the inertia ratio which increases with the non-dimensional amplitude and the stratum depth whose increase decreases the amplitude. For much lower inertia ratios than $\frac{16}{3}$, it is readily deduced that the stratum depth affects both resonant frequency factor and amplitude in the combined sense that the stiffness component of the function P becomes significant and that, in the region of frequency factor that makes $P = 0$, the amplitude is governed completely by the damping function Q.

The foregoing discussion is further reinforced by comparing the theoretical results for the case of an inertia ratio of 0·785 (which is much lower than $\frac{16}{3}$) on a stratum of depth 0·97 times the oscillator radius with the experimental results due to Arnold *et al.* [3] as shown in Fig. 3. Comparing this curve with the corresponding curve of $\tilde{h} = 0{\cdot}97$ in Fig. 2, one observes that the effect of inertia ratio is, again, as expected—larger bodies are associated with higher nondimensional resonant amplitudes but at lower frequency factors. The degree of agreement of this curve with experimental results justifies the fairness of the approximation to the hyperbolic tangent. The theoretical curve of Arnold *et al.* [3] has been included for the purpose of comparison. The wide departure of this curve from experimental results cannot be apportioned to the neglect of thermal damping in the

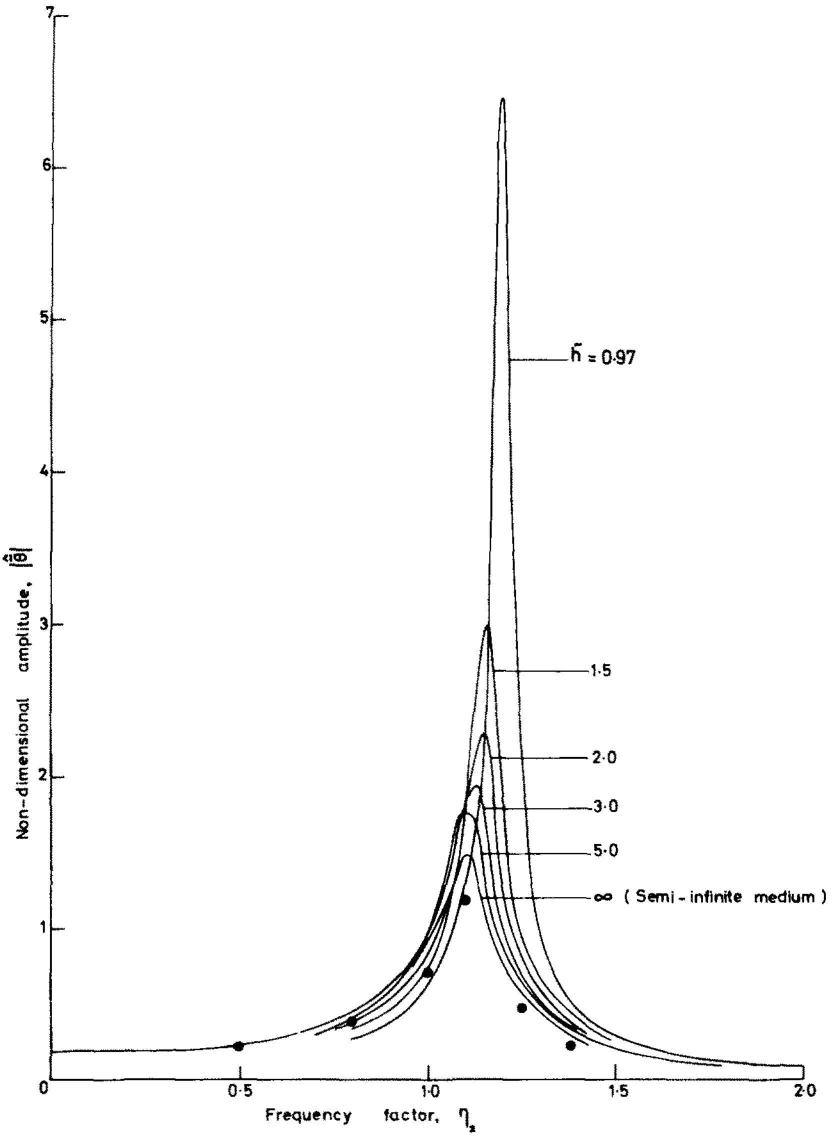

Fig. 2. Resonance curves for the torsional oscillation of a body with inertia ratio, $\bar{J}$ = 3·49 showing the effect of stratum depth. ————, Present theory; • • • • • • •, experimental results for semi-infinite medium by Arnold *et al.* [3].

fundamental assumption of the theory because the variation and nature of this thermal damping in relation to amplitude and frequency factor are not likely to be so regular as to keep the form of the resonance curve unchanged if they can be applied to correct the theoretical curve that neglects thermal damping. The more likely explanation for the departure of their theoretical curve from the experimental results is that the assumption of a static stress distribution of a semi-infinite medium as a starting point for a vibration problem on an elastic stratum as discussed in the introduction underestimates the value of

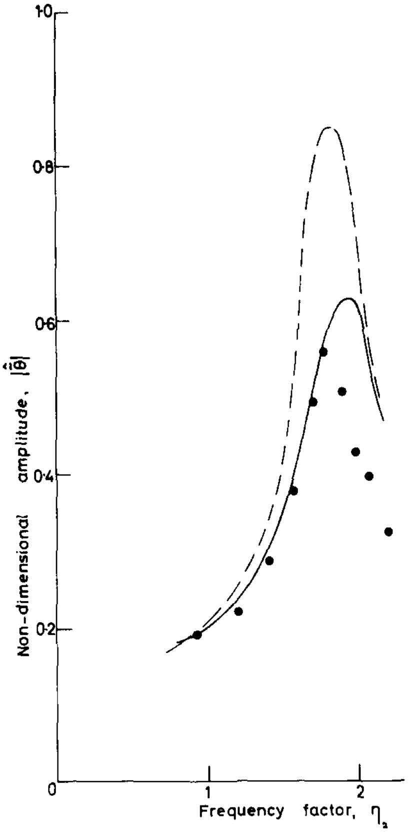

FIG. 3. Amplitude response of a rigid torsional oscillator with inertia ratio, $\bar{J}$ = 0·786 on an elastic stratum of depth $\bar{h}$ = 0·97 referred to the oscillator radius. ————, Present theory; – – – – – – –, theory of Arnold *et al.* [3]; • • • • • • •, experimental results of Arnold *et al.* [3].

dispersion damping. It is to be recalled that this damping increases with frequency factor and low inertia ratio bodies, as in this example, are more affected by damping in limiting their amplitudes.

6. CONCLUSION

The problem of the torsional oscillation of a rigid circular body on an infinitely wide elastic stratum has been formulated exactly for the first time by reducing it to the solution of dual integral equations with the Hankel transform of the dynamic stress distribution under the oscillator as the unknown. Although an exact solution of these equations is at present impossible, the author has succeeded in finding an approximate function for the kernel of the equation which makes an exact solution impossible. This approximate representation of the exact dual integral equations leads directly to a novel method of deducing the solution of the stratum problem readily from the corresponding semi-infinite medium solution. Essentially, the in-phase and quadrature components of stress under the rigid body on a semi-infinite medium are modified by defining an equivalent frequency factor which incorporates the stratum depth. This modified complex stress system is then used to obtain the resonance curves of the rigid body on an elastic stratum. Good agreement has been obtained with the experiment of Arnold *et al.* [3] and the effect of stratum depth has been determined for one inertia ratio showing how the stratum solution tends to the semi-infinite medium case of which practical results have also been included. It is found that, for this inertia ratio, a stratum depth of five times the radius of the oscillator is a fair approximation to a semi-infinite medium.

It should be noted, in conclusion, that in this approach there is no need to assume any unknown function as it is generally done but first to find an exact formulation of the problem by establishing the governing dual integral equations which has been given here for the first time.

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Абстракт—Представляется точная формулировка смешанной краевой задачи путем выведения управляемых парных интегральных уравнений для обыкновенного предполагаемого неизвестного распределения динамических напряжений в твердом теле. Ядро этих уравнений содержит гиперболический тангенс, вследствие этого точное решение является очень трудным. Тем не менее находится, что функция $X/(1+X^2)^{1/2}$ оказывается хорошим приближением для $\tanh X$ для всех действительных значений X. Введение этого приближения приводит к заключению, что упругий слой, возбужденный крутящим осциллятором для фактора частоты η_2 ведет себя приближительно как полубесконечная среда, возбужденная фактором с эквивалентной частотой $\eta_{2e} = (\eta_2{}^2-1/\bar{h}^2)^{1/2}$, где $\bar{h}$-толщина слоя, отнесенная к радиусу осциллятора и $1/\bar{h} \leq \eta_2$. Кривые резонанса для твердо го тела, основанные на предлагаемом решении, хорошо сходятся с опубликованными экспериментальными результатами.